Journal of Pure and Applied Algebra 38 (1985) 337-351
North-Holland

THE EXT-ALGEBRA OF A GOLOD RING

Gunnar SJÖDIN
Docentbacken 3, S-104 05 Stockholm, Sweden

Communicated by C. Löfwall
Received 25 June 1985

Dedicated to Jan-Erik Roos on his 50-th birthday

Editors note. This article was written nine years ago, but, regretfully, appeared only as two preprints (1976-3 and 1976-6 from University of Stockholm). The presentation is direct and self-contained. The proofs highly depend on computations in the Ext-algebra. There are other, perhaps more elegant, proofs today, but this paper gives a good insight in the theory of Golod rings. Some of the results are used by Löfwall in his contribution to this volume.

Introduction

In the following $(R, \mathfrak{m})$ denotes a local noetherian ring with maximal ideal $\mathfrak{m}$ and residue field $\mathbf{k}$. In [9] the Ext-algebra of a local complete intersection is determined. The purpose of this paper is to attack the same problem for Golod rings. It has been shown by Levin (cf. [3, p. 186]) that the Ext-algebra of a Golod ring is finitely generated. We show that it is even finitely *presented,* and, if $R=S/\Omega$, where $(S, \mathfrak{p})$ is regular and $\mathfrak{p}^{2r-3}\subset\Omega\subset\mathfrak{p}^r$ for some $r\geq 3$, then the relations can be fairly well understood. In the special case $R=S/\mathfrak{p}^r$, where $(S, \mathfrak{p})$ is regular, we calculate the Ext-algebra exactly and present the result in terms of Lie algebras. In this case the Ext-algebra is generated by its 1- and 2-dimensional elements.

Notations and conventions

We shall use the following symbols, definitions and conventions.

(1) If x is an object assigned some degree, then $|x|$ denotes this degree. In expressions for signs we even drop $|\cdot|$ so we write $(-1)^{a\cdot b}$ instead of $(-1)^{|a|\cdot|b|}$.

(2) If A is a graded algebra over some ring, then the commutator $[\cdot, \cdot]$ is defined by $[a, b]=ab-(-1)^{a\cdot b}ba$. When we study commutators within an indexed subset $\{a_i\}$ of an algebra we put $[a_i, a_i]=a_i^2$ (instead of $2a_i^2$) if a_i is of odd degree.

(3) The sign -1 in a diagram means that the diagram is anti-commutative.

(4) If X is a graded module over a ring, then TX denotes the tensor algebra of

X over this ring. If $\{x_\alpha\}$ is a set of elements given some degrees, then $\mathbf{k}\langle\{x_\alpha\}\rangle$ is the free non-commutative algebra over $\mathbf{k}$ on the x_α's; i.e., the tensor algebra of the graded vector space with $\{x_\alpha\}$ as a basis. By $\mathbf{k}[\{x_\alpha\}]$ we mean the free strictly commutative algebra on the x_α's.

(5) If X, Y are complexes over a ring A, then

$$\operatorname{Hom}_A(X, Y) = \bigoplus_n \prod_{i-j=n} \operatorname{Hom}_A(X_i, Y_j)$$

is a complex with differential as in [5, VI 7.6].

(6) Let $P \xrightarrow{\varepsilon} \mathbf{k}$ be a projective resolution of $\mathbf{k}$. Then

$$H\operatorname{Hom}_R(1, \varepsilon)\colon H\operatorname{Hom}_R(P, P) \xrightarrow{\simeq} H\operatorname{Hom}_R(P, \mathbf{k}) = \operatorname{Ext}_R(\mathbf{k}, \mathbf{k})$$

and the product in $\operatorname{Ext}_R(\mathbf{k}, \mathbf{k})$ is induced by the composite $\circ$ in $\operatorname{Hom}_R(P, P)$ under this isomorphism (cf. [5]).

1. The minimal resolution of k for a Golod ring

Let $(R, \mathfrak{m})$ be a local ring and let K be the Koszul complex on a minimal set of generators $x_1, \ldots, x_n$ of $\mathfrak{m}$. Thus we have a basis $T_1, \ldots, T_n$ of K_1 such that $dT_i = x_i$. Let $|H_i K| = c_{i+1}$ $1 \le i \le n$. We put $c = c_2 + \ldots + c_{n+1}$. Choose a set of cycles $z_1, \ldots, z_c$ in $\tilde{K}$ inducing a basis of $\tilde{H}K$. Such a set will be called a minimal set of cycles. We may assume that $|z_i| < |z_j|$ implies that $i < j$.

Definition. We say that R is a Golod ring if the set $\{z_i\}_{1 \le i \le c}$ allows Massey operations which are everywhere defined; i.e., if for every finite sequence $y_1, \ldots, y_t$, where $y_j = z_{i_j}$, there is an element $\gamma(y_1, \ldots, y_t) \in K$ of degree $|y_1| + \ldots + |y_t| + t - 1$ such that

(a) $\gamma(z_i) = z_i$,

(b) $d\gamma(y_1, \ldots, y_t) = \sum_{1 \le i \le t-1} (-1)^{y_1 + \ldots + y_i + i} \gamma(y_1, \ldots, y_i) \cdot \gamma(y_{i+1}, \ldots, y_t)$.

Note that by induction we get $\gamma(y_1, \ldots, y_t) \in \mathfrak{m}K$.

Remark. It may be shown (see e.g. [8]) that the above definition is independent of the choice of the z_i's.

Now suppose that R is a Golod ring and that the z_j's and γ are as above. Let V be a graded free R-module with $|V_i| = |H_{i-1}K| = c_i$, $2 \le i \le n+1$. Choose a basis $u_1, \ldots, u_c$ of V corresponding to the z_j's ($|u_j| = |z_j| + 1$) and define and R-linear map $\beta\colon TV \to \mathfrak{m}K \subset K$ by ($w_j = u_{i_j}, y_j = z_{i_j}$)

$$\beta(w_1 \otimes \ldots \otimes w_t) = \gamma(y_1, \ldots, y_t).$$

In particular $\beta(u_i) = z_i$ and β has degree $+1$. If $v_1, \ldots, v_t$ are basis elements of V, then

$$d\beta(v_1\otimes\ldots\otimes v_t)=d\gamma(y_1,\ldots,y_t)$$

$$=\sum_{1\le i\le t-1}(-1)^{y_1+\ldots+y_i+i}\gamma(y_1,\ldots,y_i)\cdot\gamma(y_{i+1},\ldots,y_t)$$

$$=\sum_{1\le i\le t-1}(-1)^{v_1+\ldots+v_i}\beta(v_1\otimes\ldots\otimes v_i)\cdot\beta(v_{i+1}\otimes\ldots\otimes v_t)$$

and by linear extension the same formula is true for any $v_1,\ldots,v_t\in V$.

Now define $d:K\otimes_R TV\to K\otimes_R TV$ by

$$d(\lambda\otimes v_1\otimes\cdots\otimes v_t)=d\lambda\otimes v_1\otimes\ldots\otimes v_t$$

$$+(-1)^{\lambda}\lambda\otimes\sum_{1\le i\le t}\beta(v_1\otimes\ldots\otimes v_i)\otimes v_{i+1}\otimes\ldots\otimes v_t$$

where $\lambda\in K$ and $v_j\in V$. Note that $d(K\otimes TV)\subset\mathfrak{m}(K\otimes TV)$ and that $K_0=R$, $(TV)_0=R$ give natural imbeddings of V, TV, K in $K\otimes TV$. With these imbeddings we have $d\,|\,V=\beta$ and $d\,|\,K=d_K$. Now

$$d^2(v_1\otimes\cdots\otimes v_t)=d\sum_{1\le i\le t}\beta(v_1\otimes\ldots\otimes v_i)\otimes v_{i+1}\otimes\ldots\otimes v_t$$

$$=\sum_{1\le i\le t}d\beta(v_1\otimes\ldots\otimes v_i)\otimes v_{i+1}\otimes\ldots\otimes v_t$$

$$+\sum_{1\le i<j\le t}(-1)^{v_1+\ldots v_i-1}\beta(v_1\otimes\ldots\otimes v_i)\cdot\beta(v_{i+1}\otimes\ldots\otimes v_j)\otimes v_{j+1}\otimes\ldots\otimes v_t$$

$$=\Bigg(\sum_{1\le i\le t-1}\mathrm{d}\beta(v_1\otimes\ldots\otimes v_i)\otimes v_{i+1}\otimes\ldots\otimes v_{t-1}$$

$$+\sum_{1\le i<j\le t-1}(-1)^{v_1+\ldots+v_i-1}\beta(v_1\otimes\ldots\otimes v_i)$$

$$\cdot\beta(v_{i+1}\otimes\cdots\otimes v_j)\otimes v_{j+1}\otimes\ldots\otimes v_{t-1}\Bigg)\otimes v_t+d\beta(v_1\otimes\ldots\otimes v_t)$$

$$+\sum_{1\le i\le t-1}(-1)^{v_1+\ldots v_i-1}\beta(v_1\otimes\ldots\otimes v_i)\cdot\beta(v_{i+1}\otimes\ldots\otimes v_t).$$

The sum of the last two terms is zero and hence

$$d^2(v_1\otimes\ldots\otimes v_t)=d^2(v_1\otimes\ldots\otimes v_{t-1})\otimes v_t=\ldots=d^2v_1\otimes v_2\otimes\ldots\otimes v_t=0.$$

Thus $d^2\,|\,TV=0$. Let $\lambda\in K$, $w\in TV$. Then $d(\lambda\otimes w)=d\lambda\otimes w+(-1)^{\lambda}\lambda\otimes dw$. It follows that

$$d^2(\lambda\otimes w)=d^2\lambda\otimes w+(-1)^{\lambda-1}d\lambda\otimes dw+(-1)^{\lambda}d\lambda\otimes dw+\lambda\otimes d^2w=0.$$

We have established that $K\otimes TV$ is a complex.

Finally we show (cf. [2]):

Theorem 1. *With the above definition of d, $K \otimes TV$ is a minimal resolution of* $\mathbf{k}$. *In particular the Poincaré series* $P_R(z) = (1+z)^n/(1-c_2z^2 - \ldots - c_{n+1}z^{n+1})$.

Proof. All that remains to prove is that $H_i(K \otimes TV) = 0$ for $i > 0$ and $H_0(K \otimes TV) = \mathbf{k}$. Let $F_pTV = \bigoplus_{t \le p} V_{i_1} \otimes \ldots \otimes V_{i_t}$ and let $F_p(K \otimes TV) = K \otimes F_pTV$. Then $dF_p(K \otimes TV) \subset F_p(K \otimes TV)$; i.e., $K \otimes TV$ is a regularly filtered complex and $E^0(K \otimes TV) = K \otimes TV$ with $d^0(\lambda \otimes w) = d\lambda \otimes w$ so that $E^1(K \otimes TV) = HK \otimes TV$. Now d^1 is given by

$$d^1(\{\lambda\} \otimes v_1 \otimes \ldots \otimes v_p) = \{(-1)^\lambda \lambda \cdot \beta(v_1)\} \otimes v_2 \otimes \ldots \otimes v_p.$$

In particular $d^1(\{1\} \otimes v_1 \otimes \ldots \otimes v_p) = \{\beta(v_1)\} \otimes v_2 \otimes \ldots \otimes v_p$.

However, $V \xrightarrow{\beta} ZK \longrightarrow \tilde{H}K$ maps an R-basis of V to a $\mathbf{k}$-basis of $\tilde{H}K$. It follows that

$$ZE^1(K \otimes TV) = (\tilde{H}K \otimes TV) \oplus H_0K,$$

$$BE^1(K \otimes TV) = d^1(H_0K \otimes TV) = \tilde{H}K \otimes TV,$$

and hence that $E^2(K \otimes TV) = H_0K = \mathbf{k}$, which completes the proof. □

2. Examples of Golod rings

Let (S, p) be a regular local ring. Basically the only known examples of Golod rings are:

(1) $R = S/\mathfrak{p}^r$ (cf. [2] or [3] for the equicharacteristic case),

(2) $R = S/x \cdot \Omega$, where $\Omega \subset \mathfrak{p}$ is an ideal and $x \in \mathfrak{p}$ (cf. [8]).

Obviously a ring is a Golod ring if there is a minimal set of cycles $z_1, \ldots, z_c$ such that $z_i \cdot z_j = 0$ for any i, j. This is the case in the two examples above. We are going to discuss case (1) in some detail and also exhibit a minimal set of cycles for $\tilde{K}$. Let L be the Koszul complex over S on a minimal set of generators $y_1, \ldots, y_n$ of $\mathfrak{p}$. We may assume that $r \ge 2$. Put $x_i = \bar{y}_i \in \mathfrak{m} = \mathfrak{p}/\mathfrak{p}^r$. Then $x_1, \ldots, x_n$ is a minimal set of generators of $\mathfrak{m}$ and we may take $K = L \otimes_R S = L/\mathfrak{p}^r L$ as the Koszul complex of R.

Lemma 1. *The mapping* $d: \mathfrak{p}^{s-1}L_{i+1} \to B_iL \cap \mathfrak{p}^s L_i$ *is an epimorphism for* $i \ge 0$, $s \ge 1$ *and* $B_iL \cap \mathfrak{p}^s L_i = Z_i \mathfrak{p}^s L$ *for* $i \ge 1$.

Proof. Let $a \in B_iL \cap \mathfrak{p}^s L_i$. Then $a = db$ for some $b \in L_{i+1}$. We have $b = x + y$, where we write x, y as linear combinations of the natural basis elements of L_{i+1}. For x these coefficients are polynomials of degree $\le s-2$ in the y_j's and the coefficients of these polynomials lie in $S - \mathfrak{p}$. For y the coefficients are in $\mathfrak{p}^{s-1}$. Thus $a = dy \in \mathfrak{p}^s L_i$ so that $dx \in \mathfrak{p}^s L_i$. Now dx is a linear combination of the natural basis elements of L_i and the coefficients are polynomials of degree $\le s-1$ in the y_j's. The coefficients of these polynomials lie in $S - \mathfrak{p}$. However, since S is regular $\bigoplus_{i \ge 0} \mathfrak{p}^i/\mathfrak{p}^{i+1} = \mathbf{k}[Y_1, \ldots, Y_n]$ the polynomial ring over $\mathbf{k}$ on the Y_j's, where Y_j is the image of y_j in $\mathfrak{p}/\mathfrak{p}^2$. It follows that $dx=0$ and hence that $d\colon \mathfrak{p}^{s-1}L \to B_iL \cap \mathfrak{p}^sL_i$ is an epimorphism. Furthermore, if $i \geq 1$, then $H_iL=0$ so that $Z_i\mathfrak{p}^sL = Z_iL \cap \mathfrak{p}^sL_i = B_iL \cap \mathfrak{p}^sL$. □

Lemma 2. *We have* $Z_iK = B_iK + \mathfrak{m}^{r-1}K_i$ *for* $i \geq 1$. *In particular we may choose a minimal set of cycles in* $\mathfrak{m}^{r-1}K$ *and hence* R *is a Golod ring.*

Proof. Let $\bar{x} \in Z_iK$, where $x \in L_i$. Then $dx \in \mathfrak{p}^rL_{i-1}$. Thus, according to Lemma 1, $dx = dy$ for some $y \in \mathfrak{p}^{r-1}L_i$. It follows that $x - y \in Z_iL = B_iL$ and hence $\bar{x} = \overline{x-y} + \bar{y}$, where $\overline{x-y} \in B_iK$ and $\bar{y} \in \mathfrak{m}^{r-1}K_i$. □

Definition. Let $1 \leq i \leq n$ and $r \leq 2$. Then we put

$$c_{i,r,n} = |H_iK| \qquad (\dim_k(\mathfrak{p}/\mathfrak{p}^2) = n,\ R = S/\mathfrak{p}^r),$$

$$d_{i,r,n} = \binom{r+i-2}{r-1}\binom{r+n-1}{r+i-1},$$

$$e_{i,r,n} = \binom{r+i-2}{r-1}\binom{i-1}{i-1} + \binom{r+i-1}{r-1}\binom{i}{i-1} + \cdots + \binom{r+n-2}{r-1}\binom{n-1}{i-1}.$$

For the following result, cf. [2].

Lemma 3. We have $c_{i,r,n} = d_{i,r,n} = e_{i,r,n}$.

Proof. Consider the exact sequence

$$0 \to \mathfrak{p}^rL \to L \to K \to 0$$

of complexes over S. Since $H_iL = 0$ for $i \neq 0$ and $H_0L \xrightarrow{\simeq} H_0K$, the corresponding long exact sequence shows that $H_iK \simeq H_{i-1}\mathfrak{p}^rL$ for $i \geq 1$. We also have an exact sequence

$$0 \to \mathfrak{p}^rL \to \mathfrak{p}^{r-1}L \to \mathfrak{p}^{r-1}L/\mathfrak{p}^rL \to 0. \tag{1}$$

Note that the differential of $\mathfrak{p}^{r-1}L/\mathfrak{p}^rL$ is zero. Now $d\colon \mathfrak{p}^{r-1}L_i \to Z_{i-1}\mathfrak{p}^rL$ is an epimorphism for $i \geq 2$ and hence so is $\delta\colon \mathfrak{p}^{r-1}L_i/\mathfrak{p}^rL_i \to H_{i-1}\mathfrak{p}^rL$ in the long exact sequence of (1). In the same sequence $H_0\mathfrak{p}^{r-1}L \xrightarrow{\simeq} \mathfrak{p}^{r-1}/\mathfrak{p}^r$. It follows that

$$0 \to H_i\mathfrak{p}^{r-1}L \to \mathfrak{p}^{r-1}L_i/\mathfrak{p}^rL_i \to H_{i-1}\mathfrak{p}^rL \to 0$$

is exact for $i \geq 1$. Thus

$$c_{i,r,n} + c_{i+1,r-1,n} = |\mathfrak{p}^{r-1}L_i/\mathfrak{p}^rL_i| = \binom{r+n-2}{n-1}\binom{n}{i}$$

for $1 \leq i \leq n-1, r \geq 3$. On the other hand the same relation is true for $d_{i,r,n}$ and

$$c_{1,r,n} = |H_0\mathfrak{p}^rL| = |\mathfrak{p}^r/\mathfrak{p}^{r+1}| = \binom{r+n-1}{n-1} = \binom{r+n-1}{r} = d_{1,r,n}$$

for $1 \le n$, $r \ge 2$. By induction $c_{i,r,n} = d_{i,r,n}$ for $1 \le i \le n$, $r \ge 2$. Now

$$e_{i,r,n+1} = e_{i,r,n} + \binom{r+n-1}{r-1}\binom{n}{i-1}$$

for $1 \le i \le n$, $r \ge 2$ and the same formula is true for $d_{i,r,n}$. Furthermore,

$$e_{i,r,i} = \binom{r+i-2}{r-1} = d_{i,r,i}$$

for $1 \le i$, $r \ge 2$ and hence $e_{i,r,n} = d_{i,r,n}$ $1 \le i \le n$, $r \ge 2$. □

Let A_i be the set of subsequences $a = (a_1, \dots, a_i)$ of $(1, \dots, n)$ of length i and let Q be the set of sequences $q = (q_1, \dots, q_n)$ of non-negative integers such that $q_1 + \dots + q_n = r-1$. let $C_i = \{(a, q) \in A_i \times Q \mid q_j = 0 \text{ for } j > a_i\}$. For $(a, q) \in A_i \times Q$ we put $x_{a,q} = x_1^{q_1} \dots x_n^{q_n} T_{a_1} \dots T_{a_i}$.

Lemma 4. *The set* $\{x_{a,q}\}_{(a,q) \in C_i}$ *is a minimal set of cycles in* K_i *for* $i \ge 1$.

Proof. Obviously $x_{a,q} \in Z_i K$ for $(a, q) \in C_i$. Let $(a_1, \dots, a_{i+1}) \in A_{i+1}$. Then

$$d(T_{a_1} \dots T_{a_{i+1}}) = \sum_{1 \le t \le i+1} (-1)^{t-1} x_{a_t} T_{a_1} \dots \hat{T}_{a_t} \dots T_{a_{i+1}}$$

and hence

$$x_{a_{i+1}} T_{a_1} \dots T_{a_i} \equiv \sum_{1 \le t \le i} (-1)^t x_{a_t} T_{a_1} \dots \hat{T}_{a_t} \dots T_{a_{i+1}} \pmod{B_i K}$$

from which it easily follows that $\{x_{a,q}\}_{(a,q) \in C_i}$ generates $Z_i K \pmod{B_i K}$. On the other hand the number of elements in this set is $e_{i,r,n} = |H_i K|$ since

$$\binom{r+i-2+s}{r-1}\binom{i-1+s}{i-1} = \binom{r-1+i+s-1}{i+s-1}\binom{i-1+s}{i-1},$$

which is the number of $x_{a,q}$'s with $a_i = i+s$. We conclude that the set is a minimal set of cycles in K_i. □

3. The Ext-algebra of a Golod ring

We keep the notations of Section 1. In this section we want to prove the following result.

Theorem 2. *Let R be a Golod ring. Then*

$$\operatorname{Ext}_R(\mathbf{k}, \mathbf{k}) = \mathbf{k}\langle X_1, \dots, X_n, Y_1, \dots, Y_c \rangle / ([X_i, X_j] - \Phi_{i,j}, 1 \le i \le j \le n,$$
$$[X_i, Y_j] - \Psi_{i,j}, 1 \le i \le n, 1 \le j \le c)$$

where $|X_i|=1$, $1\le i\le n$, *and* $|Y_j|=|z_j|+1$. *The* $\Phi_{i,j}$*'s and* $\Psi_{i,j}$*'s are polynomials in the* Y_t*'s. In particular the Ext-algebra of a Golod ring is finitely presented.*

The proof of this theorem will be contained in the following lemmas. Let $P=K\otimes TV$. Note that P has the structure of a differential graded module over the differential graded algebra K.

Lemma 5. (a) *Let* $f\in \mathrm{Hom}_R(K_1,R)$. *Then there is an* $F\in Z^1\mathrm{Hom}_R(P,P)$ *such that* $F\,|\,K_1=f$ *and* $F(\lambda x)=F(\lambda)\cdot x+(-1)^{\lambda}\lambda F(x)$ *for* $\lambda\in K$, $x\in P$.

(b) *Let* $g\in\mathrm{Hom}_R(V,R)$. *Define* $G\in\mathrm{Hom}_R(P,P)$ *by*

$$G(\lambda\otimes y_1\otimes\cdots\otimes y_t)=(-1)^{y_t\cdot(\lambda+y_1+\cdots y_{t-1})}g(y_t)\lambda\otimes y_1\otimes\cdots\otimes y_{t-1}$$

for $\lambda\in K$, $y_j\in V$ (*in particular* $G\,|\,K=0$). *Then* $G\in Z^{|G|}\mathrm{Hom}_R(P,P)$, $G\,|\,V=g$ *and* $G(\lambda x)=(-1)^{\lambda\cdot G}G(x)$ *for* $\lambda\in K$, $x\in P$.

Proof. (b) By direct checking.

(a) Define F on K by putting

$$F(T_{a_1}\cdots T_{a_t})=\sum_{1\le j\le t}(-1)^{j-1}f(T_{a_j})T_{a_1}\cdots\hat{T}_{a_j}\cdots T_{a_t}.$$

Next define $F_s:P_s\to P_{s-1}$ by induction on s as follows. We already have defined F_1. Let $s\ge 2$ and suppose that F_i, $1\le i\le s-1$, has been defined satisfying the requirements. Now choose F_s: $(TV)_s\to P_{s-1}$ such that

$$\begin{array}{ccc} (TV)_s & \xrightarrow{\;d\;} & P_{s-1}\\ F_s\big\downarrow & -1 & \big\downarrow F_{s-1}\\ P_{s-1} & \xrightarrow[\;d\;]{} & P_{s-2}\end{array}$$

which is possible since $dF_{s-1}d=-F_{s-2}dd=0$ and $(TV)_s$ is free over R. Then define $F_s:K_i\otimes(TV)_{s-i}\to P_{s-1}$ for $i\ge 1$ by $F_s(\lambda\otimes x)=F(\lambda)\otimes x+(-1)^{\lambda}\lambda\otimes F(x)$ for $\lambda\in K_i$, $x\in(TV)_{s-i}$. Then, using the induction hypothesis and the fact that the differential on P satisfies $d(\mu y)=(d\mu)y+(-1)^{\mu}\mu dy$ for $\mu\in K$, $y\in P$, we get $dF_s(\lambda\otimes x)=F_{s-1}d(\lambda\otimes x)$ so that we may continue the induction to get an $F\in Z^1\mathrm{Hom}_R(P,P)$ with $F\,|\,K_1=f$. Note that if the inductive construction is made as above then $F(\lambda x)=F(\lambda)x+(-1)^{\lambda}\lambda F(x)$ for $\lambda\in K$, $x\in P$. This completes the proof. □

Definition. Let $f_i\in\mathrm{Hom}_R(K_1,R)$, $1\le i\le n$, be given by $f_i(T_j)=\delta_{i,j}$ and $g_i\in\mathrm{Hom}_R(V,R)$, $1\le i\le c$, by $g_i(u_j)=\delta_{i,j}$. Let $F_i, G_j\in Z\mathrm{Hom}_R(P,P)$ correspond to f_i, g_j as in Lemma 5. Then we put

$$X_i:P\xrightarrow{F_i}P\xrightarrow{\varepsilon}\mathbf{k},$$

$$Y_j : P \xrightarrow{\mathbf{G_j}} P \xrightarrow{\varepsilon} \mathbf{k}.$$

Thus $X_i \in \mathrm{Ext}^1_R(\mathbf{k},\mathbf{k})$ and $Y_j \in \mathrm{Ext}^{|c_j|+1}_R(\mathbf{k},\mathbf{k})$.

The following result should be compared with [3, p. 186].

Lemma 6. *The X_i's and Y_j's together generate* $\mathrm{Ext}_R(\mathbf{k},\mathbf{k})$.

Proof. Let A be the set of subsequences of $(1,\dots,n)$ and let B be the set of finite sequences taking values in $\{1,\dots,c\}$. Order A by $a=(a_1,\dots,a_s)<(a_1',\dots,a_t')=a'$ if $s<t$ or if $s=t$ and the last non-vanishing $a_i-a_i'<0$. Order B in the same way and $A\times B$ by $(a,b)<(a',b')$ if $b<b'$ or $b=b'$ and $a<a'$. For $a=(a_1,\dots,a_s)\in A$ we put

$$X^a = X_{a_1}\cdots X_{a_s},\quad T^a=T_{a_1}\cdots T_{a_s}\qquad (=1 \text{ in both cases if } a \text{ is empty})$$

and for $b=(b_1,\dots,b_s)\in B$ we put

$$Y^b = Y_{b_1}\cdots Y_{b_s},\quad U^b=u_{b_1}\cdots u_{b_s}\qquad (=1 \text{ in both cases if } b \text{ is empty}).$$

Then $\{T^aU^b\}_{(a,b)\in A\times B}$ is a well-ordered basis of P. As is easily checked

$$X^aY^b(T^{a'}U^{b'}) = \begin{cases} \pm 1 & \text{if } (a,b)=(a',b'),\\ 0 & \text{if } (a,b)>(a',b'). \end{cases}$$

Thus $\{X^aY^b\}_{(a,b)\in A\times B}$ is expressed by a triangular matrix, with ± 1's in the diagonal, in the $\mathbf{k}$-basis of $\mathrm{Ext}_R(\mathbf{k},\mathbf{k})=\mathrm{Hom}_R(P,\mathbf{k})$ dual to $\{T^aU^b\}_{(a,b)\in A\times B}$ and hence $\{X^aY^b\}_{(a,b)\in A\times B}$ is itself a basis of $\mathrm{Ext}_R(\mathbf{k},\mathbf{k})$. In particular the X_i's and Y_j's generate $\mathrm{Ext}_R(\mathbf{k},\mathbf{k})$ as an algebra. □

Lemma 7. *Let $f\in \mathrm{Hom}_R(K_1,\mathbf{k})$, $g\in\mathrm{Hom}_R(V_s,\mathbf{k})\subset\mathrm{Hom}_R(V,\mathbf{k})$. Then $[G,F]$ annihilates the elements of P_{s+1} which lie in $K_i\otimes TV$ for $i\geq 1$; i.e., everything but possibly $(TV)_{s+1}$, where it coincides with $G\circ F$.*

Proof. Let $(TV)^t=\bigoplus V_{i_1}\otimes\cdots\otimes V_{i_t}$. It follows from the definition of G that it annihilates everything in P_{s+1} but possibly $K_1\otimes V_s$ (note that $V_1=0$). Hence the same is true for $F\circ G$. Let $\lambda\otimes x=\lambda x\in P_{s+1}$ where $\lambda\in K_i$, $x\in(TV)^t$. Then $F(\lambda x)=F(\lambda)x+(-1)^\lambda\lambda F(x)$ and thus

$$G\circ F(\lambda x)=(-1)^{(\lambda-1)\cdot G}F(\lambda)\cdot G(x)+(-1)^{\lambda+\lambda\cdot G}\lambda\cdot(G\circ F)(x).$$

If $i\geq 1$, then $|x|\leq s$ so $(G\circ F)(x)=0$. If $i\geq 1$ and $t\geq 2$, then $G(x)=0$ by the definition of G. If $i\geq 2$, then $|x|<s$ whence $G\circ F(\lambda x)=0$. Thus $G\circ F(\lambda x)=0$ but possibly when $i\leq 1$ and $t=1$; i.e., $G\circ F$ annihilates everything but possibly $K_1\otimes V_s$ and $(TV)_{s+1}$. Let $i=t=1$. Then $G\circ F(\lambda x)=F(\lambda)\cdot G(x)=f(\lambda)\cdot g(x)$ whereas $F\circ G(\lambda x)=F((-1)^{\lambda\cdot G}\lambda\cdot G(x))=(-1)^{F\cdot G}f(\lambda)\cdot g(x)$; i.e., $[F,G](\lambda x)=0$ and hence $[F,G]$ is zero on everything in P_{s+1} but possibly $(TV)_{s+1}$. □

Now $\mathrm{Hom}_R(1,\varepsilon): Z\mathrm{Hom}_R(P,P) \to \mathrm{Hom}_R(P,\mathbf{k}) = \mathrm{Ext}_R(\mathbf{k},\mathbf{k})$ is an epimorphism of algebras (cf. [9]). Thus, with the notations of Lemma 7, $[\varepsilon \circ F, \varepsilon \circ G] = \varepsilon \circ [F,G]$ is zero on everything in P_{s+1} but $(TV)_{s+1}$. However, every element in $\mathrm{Ext}_R^{s+1}(\mathbf{k},\mathbf{k})$ with this property is a polynomial in the Y_j's since the polynomials in these variables of algebra-degree $s+1$ must annihilate $K_i \otimes (TV)_{s+1-i}$ for $i \geq 1$. Thus we get

Lemma 8. *The commutators $[X_i, Y_j] = \Psi_{i,j}$ are polynomials in the Y_t's.*

Lemma 9. *The commutators $[X_i, X_j] = \Phi_{i,j}$ are linear combinations of $Y_1, \ldots, Y_{c_2}$.*

Proof. It suffices to show that $[F_i, F_j](K_2) = 0$ and this follows directly from the construction of the F_i's on K. $\square$

The Lemmas 6, 8 and 9 now show that there is an algebra epimorphism from the algebra Λ on the right hand side of the formula in Theorem 2 to $\mathrm{Ext}_R(\mathbf{k},\mathbf{k})$. Thus to complete the proof of Theorem 2 it only remains to show that

$$H_\Lambda \leq H_{\mathrm{Ext}_R(\mathbf{k},\mathbf{k})} = H_{\mathbf{k}[X_1,\ldots,X_n]} \cdot H_{\mathbf{k}\langle Y_1,\ldots,Y_c\rangle}$$

where $H_\Lambda = H_\Lambda(z) = \sum_{i\geq 0} \dim_{\mathbf{k}} \Lambda_i z^i$ is the Hilbert series of Λ, and $\leq$ denotes coefficient-wise inequality. This is a consequence of the following lemma.

Lemma 10. *Let C be a connected graded algebra of finite type over a field k and and let $A = \{a_i\}$, $B = \{b_j\}$ be two finite sets of elements in C of positive degree such that $A \cup B$ generates C as an algebra. Suppose that $[a_i, a_j], [a_i, b_j]$ are polynomials in the b_j's. Then $H_C \leq H_{\mathbf{k}[A]} \cdot H_{\mathbf{k}\langle B\rangle}$.*

Proof. Let F_pC be the set of elements in C that may be written as a polynomial in the elements of $A \cup B$ such that the polynomial degree is $\leq p$ in the elements of A. Then the elements $\bar{a}_i \in E^0_{1,\cdot}C$ and $\bar{b}_j \in E^0_{0,\cdot}C$ generate E^0C as an algebra and the $\bar{a}_i$'s strictly commute with each other and with the $\bar{b}_j$'s. Hence we get an epimorphism of algebras

$$\nu : \mathbf{k}[A] \otimes \mathbf{k}\langle B\rangle \to E^0C$$

where $\nu(a_i) = \bar{a}_i$, $\nu(b_j) = \bar{b}_j$. It follows that $H_C = H_{E^0C} \leq H_{\mathbf{k}[A]\otimes\mathbf{k}\langle B\rangle} = H_{\mathbf{k}[A]} \cdot H_{\mathbf{k}\langle B\rangle}$. $\square$

4. The Ext-algebra of $R = S/\mathfrak{p}^r$

We keep the notations of Section 2. For $(a,q) \in C_i$ we let $u_{a,q} \in V_{i+1}$ be the basis element corresponding to $x_{a,q}$, and we define $g_{a,q} \in \mathrm{Hom}_R(V,R)$ by $g_{a,q}(u_{a',q'}) = \delta_{a,q}^{a',q'}$. Let $G_{a,q} \in \mathrm{Hom}_R(P,P)$ correspond to $g_{a,q}$ as in Lemma 5 and let $Y_{a,q} = \varepsilon \circ G_{a,q}$. Then according to Lemma 6 the X_j's and $Y_{a,q}$'s generate $\mathrm{Ext}_R(\mathbf{k},\mathbf{k})$. For $a = (a_1, \ldots, a_i) \in A_i$ we put $a' = (a_2, \ldots, a_i)$. We order A_i by $a < b$ if the first non-vanishing $a_j - b_j < 0$ and Q in the same way. We order C_i by $(a,q)<(b,t)$ if $a<b$ or if $a=b$ and $q<t$.

Lemma 11. *Let $i \geq 2$ and let $(a,q) \in C_i$. Then*

$$[Y_{a',q}, X_{a_1}] = -Y_{a,q} + \text{a linear combination of } Y_{b,t}\text{'s of degree } i+1 \text{ and with } (a,q)>(b,t) + \text{a polynomial in the } Y_{b,t}\text{'s of degree less than } i+1.$$

Proof. It suffices to show that

$$[G_{a',q}, F_{a_1}]u_{b,t} = \begin{cases} -1 & \text{if } (a,q)=(b,t), \\ 0 & \text{if } (a,q)<(b,t). \end{cases}$$

Let $F=F_{a_1}$, $G=G_{a',q}$ and consider

$$\begin{array}{ccc} V_{i+1} & \xrightarrow{\quad d \quad} & K_i \\ {\scriptstyle F}\downarrow & -1 & \downarrow{\scriptstyle F} \\ V_i + K_i & \xrightarrow[\quad d \quad]{} & K_{i-1} \end{array}$$

We have

$$F(du_{b,t}) = Fx_{b,t} = F_{a_1} x_1^{t_1} \cdots x_{b_i}^{t_i} T_{b_1} \cdots T_{b_i} = \begin{cases} x_{b',t} & \text{if } a_1 = b_1, \\ 0 & \text{if } a_1 < b_1. \end{cases}$$

Thus we may choose

$$Fu_{b,t} = \begin{cases} -u_{b',t} & \text{if } a_1 = b_1, \\ 0 & \text{if } a_1 < b_1. \end{cases}$$

and hence

$$G \circ F(u_{b,t}) = \begin{cases} -1 & \text{if } (a,q)=(b,t), \\ 0 & \text{if } a_1 = b_1 \text{ and } (a,q) \neq (b,t), \\ 0 & \text{if } a_1 < b_1. \end{cases}$$

If $(a,q)<(b,t)$, then $a_1 \leq b_1$ and consequently

$$[G,F]u_{b,t} = G \circ F(u_{b,t}) = \begin{cases} -1 & \text{if } (a,q)=(b,t), \\ 0 & \text{if } (a,b)<(b,t). \end{cases} \qquad \square$$

Theorem 3. *Let $R = S/\mathfrak{p}^r$. Then* $\operatorname{Ext}_R(\mathbf{k},\mathbf{k})$ *is generated by* $\operatorname{Ext}_R^1(\mathbf{k},\mathbf{k})$ *and* $\operatorname{Ext}_R^2(\mathbf{k},\mathbf{k})$.

Proof. Using Lemma 11 and the usual argument with triangular matrices we see that the Y_t's of degree i, where $i \geq 3$, may be written as polynomials in the X_j's and the Y_j's of degree $\leq i-1$.

Remark. It is not always true for a Golod ring that the Ext-algebra is generated by its 1- and 2-dimensional elements. An example is provided by the Shamash-type (cf. [8]) ring $\mathbf{k}[[x,y]]/(x^2y, y^3)$. Then

$$\operatorname{Ext}_R(\mathbf{k},\mathbf{k}) = \mathbf{k}\langle X_1, X_2, Y_1, Y_2, Z\rangle / [X_i, X_j] = 0,$$
$$[X_i, Y_j] = 0, [X_1, Z] = 0, [X_2, Z] = [Y_1, Y_2]$$

where X_1, X_2 have degree 1, Y_1, Y_2 degree 2 and Z degree 3. It is easy to see that Z is not generated by elements of lower degree. Note also that this ring provides an example for which the Ext-algebra has non-empty centre and the ring is not a complete intersection.

5. The Y_j's are in the Lie algebra underlying $\operatorname{Ext}_R(\mathbf{k},\mathbf{k})$

In the following, E stands for $\operatorname{Ext}_R(\mathbf{k},\mathbf{k})$ and $\tilde{P}E$ for the underlying Lie algebra, cf. [9]. (*Editors note:* Today we often write $\pi(R)$ for $\tilde{P}E$.)

Let $\hat{T}V$ denote the shuffle algebra on V. Then, as in [1], $\hat{T}V$ is a Hopf Γ-algebra and hence so is $K\otimes\hat{T}V$. The author is indebted to Clas Löfwall, who helped him to finish the proof of the following lemma.

Lemma 12. *It is possible to choose* $\beta: TV \to K$ (*cf. Section* 1) *such that* $K\otimes\hat{T}V$ *becomes a differential graded algebra with divided powers.*

Proof. See [4, Proposition 2.4]. □

Let Y be the graded vector space generated by the Y_j's and let LY be the free adjusted Lie algebra on Y (cf. [10]). Now $I(K\otimes\hat{T}V\otimes\mathbf{k}) = D(K\otimes\hat{T}V\otimes\mathbf{k})\oplus U$, where $V\otimes\mathbf{k}\subset U$. It follows that $Y\subset\tilde{P}E$ (cf. [9] or [10]). Thus there is a homomorphism $LY\to\tilde{P}E$ in LIE, the category of connected adjusted Lie algebras. It follows easily from Theorem 2 that $TY\to E$ is a monomorphism and hence so is $LY\to\tilde{P}E$. Let E^1 be the trivial adjusted Lie algebra with $\operatorname{Ext}^1_R(\mathbf{k},\mathbf{k})$ as underlying vector space. Then there is an epimorphism $\tilde{P}E\to E^1$ in LIE and LY is the kernel of this map. Now $\hat{H}_{\tilde{P}E} = H_E = H_{E^1}\cdot H_{TY} = \hat{H}_{E^1}\cdot\hat{H}_{LY} = \hat{H}_{E^1\oplus LY}$ (see [10] for the definition of $\hat{H}$) and hence $H_{\tilde{P}E} = H_{E^1\oplus LY}$. It follows that

$$0\to LY\to\tilde{P}E\to E^1\to 0$$

is an exact sequence in LIE. Thus $LY = \bigoplus_{n\geq 2}\tilde{P}_nE$ and it follows that the $[X_i, X_j]$'s and $[X_i, Y_j]$'s of Theorem 2 are in LY. (Of course, it was clear before that the $[X_i, X_j]$'s are even in Y.)

It also follows from Lemma 12 that $\operatorname{Tor}^R(\mathbf{k},\mathbf{k}) = E(\operatorname{Tor}^R_1(\mathbf{k},\mathbf{k}))\otimes_{\mathbf{k}} T(s\tilde{H}K)$ as Γ-algebras. However, this is most easily seen by the fact that both sides are Hopf Γ-algebras, and thus free as Γ-algebras, and since they have the same Hilbert series these Γ-algebras must be isomorphic.

6. Special Golod rings

We say that R is a special Golod ring if there is a minimal set of cycles $z_1, \dots, z_c$ in $\tilde{K}$, such that $z_i \cdot z_j = 0$ for any i, j. Then

$$d(\lambda \otimes v_1 \otimes \cdots \otimes v_t) = d\lambda \otimes v_1 \otimes \cdots \otimes v_t + (-1)^{\lambda} \lambda \cdot \beta(v_1) \otimes v_2 \otimes \cdots \otimes v_t$$

Note that in this case Lemma 12 is trivially true.

Let $f \in \mathrm{Hom}_R(K_1, R)$ and let F have the same meaning as in Lemma 5. We start by choosing $V \xrightarrow{F} V + K$ such that

$$\begin{array}{ccc} V & \xrightarrow{\quad d \quad} & K \\ F \Big\downarrow & -1 & \Big\downarrow F \\ V+K & \xrightarrow[\quad d \quad]{} & K \end{array}$$

Then we may write $F(v) = F'(v) + F''(v)$, where $F'(v) \in V$, $F''(v) \in K$. Define $\bar{F}: K \otimes TV \to K \otimes TV$ by

$$\bar{F}(\lambda \otimes v_1 \otimes \cdots \otimes v_t) = F(\lambda) \otimes v_1 \otimes \cdots \otimes v_t + (-1)^{\lambda} \lambda \cdot F''(v_1) \otimes v_2 \otimes \cdots \otimes v_t$$
$$+ (-1)^{\lambda} \lambda \otimes \sum_{1 \le i \le t} v_1 \otimes \cdots \otimes v_{i-1} \otimes F'(v_i) \otimes v_{i+1} \otimes \cdots \otimes v_t.$$

By straightforward computation we get

$$(d\bar{F} + \bar{F}d)(v_1 \otimes \cdots \otimes v_t) = (-1)^{v_1 - 1}(F''v_1 \cdot dv_2 + dv_1 \cdot F''v_2) \otimes v_3 \otimes \cdots \otimes v_t.$$

Thus $\bar{F}$ will do as F on TV and (therefore on $K \otimes TV$ according to the proof of Lemma 5) if $dV \cdot F''V = 0$; i.e., if (the minimal cycles) $\cdot F''V = 0$.

We want to establish this property for a class of Golod rings that contains the rings of type $R = S/\mathfrak{p}^r$ for $r \ge 3$. First we need a lemma.

Lemma 13. *Let $R = S/\Omega$, where $(S, \mathfrak{p})$ is a regular local ring and $\Omega \subset \mathfrak{p}^r$ for some $r \ge 2$. Then $BK \cap \mathfrak{m}^s K = d(\mathfrak{m}^{s-1}K) = \mathfrak{m}^{s-1}BK$ if $s \le r$.*

Proof. Let L be the Koszul complex of S. Then $K = L/\Omega L$. Let $\bar{x} \in BK \cap \mathfrak{m}^s K$, where $x \in \mathfrak{p}^s L$ and $\bar{x}$ is its image in K. Then $\bar{x} = d\bar{y}$ and hence $dy - x \in \Omega L \cap \mathfrak{p}^r L$ and since $s \le r$ it follows that $dy \in \mathfrak{p}^s L$ so that $dy \in BL \cap \mathfrak{p}^s L = d(\mathfrak{p}^{s-1}L)$ by Lemma 1. Thus $dy = dz$ for some $z \in \mathfrak{p}^{s-1}L$ and we get $\bar{x} = d\bar{y} = d\bar{z} \in \mathfrak{m}^{s-1}BK$. □

Remark. If we put $r = 2$ we get Serre's well-known result about the Koszul complex (cf. [7, p. IV-50]).

If Ω above satisfies $\mathfrak{p}^{2r-2} \subset \Omega \subset \mathfrak{p}^r$ one proves exactly in the same way as for $S/\mathfrak{p}^r$ that R is a special Golod ring. This was first noticed by Löfwall (cf. [4]).

Lemma 14. *Let $R=S/\Omega$, where $(S,\mathfrak{p})$ is a regular local ring and $\mathfrak{p}^{2r-3}\subset\Omega\subset\mathfrak{p}^r$ for some $r\geq 3$. Then we may choose F'' such that $dV\cdot F''V=0$.*

Proof. We may assume that $dV\subset\mathfrak{m}^{r-1}K$. Let x be a basis element of V. Then $Fdx\in ZK\cap\mathfrak{m}^{r-1}K$ and $Fdx=du+a$, where $u\in V$ and $a\in BK$. Now $du\in\mathfrak{m}^{r-1}K$ and hence $a\in BK\cap\mathfrak{m}^{r-1}K=d(\mathfrak{m}^{r-2}K)$. Consequently $a=db$ for some $b\in\mathfrak{m}^{r-2}K$ and we may choose $F''x=-b$. Therefore we can pick an F'' such that $F''V\subset\mathfrak{m}^{r-2}K$. It follows that $dV\cdot F''V\subset\mathfrak{m}^{r-1}K\mathfrak{m}^{r-2}K=\mathfrak{m}^{2r-3}K=0$. □

Theorem 4. *Let R be as in Lemma 14. Then $[X_i,Y_j]$ is a linear combination of the Y_q's; that is*

$$[\cdot,\cdot]:E^1\otimes Y\to Y.$$

Proof. Let F,G correspond to X_i,Y_j as in Section 3. We may assume that F'' is chosen as in Lemma 14 and hence that $F=\bar{F}$. Thus, if $t\geq 2$

$$G\circ F(TV)^t_{|G|+1}\subset G(TV)^t_{|G|}=0$$

by the very definition of G. This combined with Lemma 7 completes the proof. □

Remark 1. The exponent $2r-3$ cannot be improved to $2r-2$ as shown by $R=\mathbf{k}[[x,y]]/(x^3,y^3,x^2y^2)$ $(r=3,\ 2r-2=4)$. Then

$$\begin{aligned}\operatorname{Ext}_R(\mathbf{k},\mathbf{k})=\mathbf{k}\langle X_1,X_2,Y_1,Y_2,Y_3,Z_1,Z_2\rangle/[X_i,X_j]=0,[Y_j,X_i]=0,j=1,2,\\ [Y_3,X_1]=-Z_1,[Y_3,X_2]=Z_2,[Z_1,X_1]=0,[Z_2,X_2]=0,\\ [Z_2,X_1]=[Z_1,X_2]=[Y_2,Y_1]\end{aligned}$$

where the Y_i's and Z_j's have degree 2 and 3 respectively.

Remark 2. Also the ring in Remark after Theorem 3 has the property that the conclusion of Theorem 4 is false.

Remark 3. Let, for $L\in\mathrm{LIE}$, $\operatorname{Der}L$ be the derivations of L, in a graded sense, that furthermore satisfy $f(\kappa x)=[f(x),x]$ (see [10]). Then $\operatorname{Der}L$ is a sub adjusted Lie algebra of $\operatorname{End}_{\mathbf{k}}L$. Suppose that $L',L''\in\mathrm{LIE}$ and that $\phi:L''\to\operatorname{Der}L'$ is a homomorphism in LIE. Then we define two structure maps on $L'\oplus L''$ as follows

$$[(x',x''),(y',y'')]=([x',y']+\phi(x'')(y')-(-1)^{x'\cdot y''}\phi(y'')(x'),[x''y'']),$$

$$\kappa(x',x'')=(\kappa x'+\phi(x'')(x'),x'')\quad\text{for }(x',x'')\text{ of odd degree,}$$

and with this structure we denote it by $L'\oplus_\phi L''$, the semi-direct product of L' and L'' via ϕ. Every $f\in\operatorname{Der}L'$ may be uniquely extended to $\bar{f}\in\operatorname{Der}WL'$ (WL', the enveloping algebra of L', treated as an associative algebra) and this gives us a map

$$L''\xrightarrow{\phi}\operatorname{Der}L'\to\operatorname{Der}WL'\to\operatorname{End}_{\mathbf{k}}WL'$$

in LIE and hence an algebra homomorphism

$$\tilde{\phi}: WL'' \to \mathrm{End}_{\mathbf{k}}\, WL'.$$

Thus WL' is a left WL''-module, and using $\phi(L'') \subset \mathrm{Der}\, L'$ one checks that the multiplication map $WL' \otimes WL' \to WL'$ is a map of left WL''-modules, and thus we may form the semi-tensor product $WL' \odot_{\tilde{\phi}} WL''$ (cf. [6]). It is not hard to check that $L' \oplus_{\phi} L'' \in \mathrm{LIE}$ and that $W(L' \oplus_{\phi} L'') = WL' \odot_{\tilde{\phi}} WL''$. Let $0 \to L' \to L \to L'' \to 0$ be an exact sequence in LIE. Then, as in the ordinary ungraded case, we see that this sequence splits to the right if and only if L is the semi-direct product of L' and L''. (Note that we have an extra Jacobi identity for adjusted Lie algebras, namely $[x,[x,z]] + [z, \kappa x] = 0$ for x of odd degree.) Now assume that $\psi: L \to \mathrm{End}_{\mathbf{k}}\, X$ is a representation of $L \in \mathrm{LIE}$ on X, a graded vector space. Every $f \in \mathrm{End}_{\mathbf{k}}\, X$ can be uniquely extended to $\bar{f} \in \mathrm{Der}\, LX$. From this we see that ψ induces a map $\bar{\psi} \to \mathrm{Der}\, LX$ in LIE so we may define $LX \oplus_{\bar{\psi}} L$ and we get $W(LX \oplus_{\bar{\psi}} L) = TX \odot_{\tilde{\bar{\psi}}} WL$. Let R be a local ring and let $\bar{Y} = \bigoplus_{n \geq 2} \tilde{P}_n E$. Then we have an exact sequence in LIE.

$$0 \to \bar{Y} \to PE \to E^1 \to 0$$

and this splits to the right if and only if the 1-dimensional elements commute (when this happens is described in Theorem 4 of [9]). If R is a Golod ring, then $\bar{Y} = LY$, as was shown in Section 5. Let R be as in Theorem 4. Then we get a representation

$$\psi: E^1 \to \mathrm{End}_{\mathbf{k}}\, Y$$

of the trivial adjusted Lie algebra E^1 on Y given by $\psi(x) = [x, \cdot\,]$ (ψ maps the X_i's on strictly commutative endomorphisms of Y). It follows that $\tilde{P}E = LY \oplus_{\bar{\psi}} E^1$.

7. The Ext-algebra of $R = S/\mathfrak{p}^r$ (continued)

We keep the notations of Lemma 4. It is well-known that $\mathrm{Ext}_R(\mathbf{k},\mathbf{k}) = T(E^1)$ if $r=2$ (cf. [3, p. 115]). This also follows from Theorem 3 combined with Theorem 4 of [9].

In the following we assume that $r \geq 3$. It may be shown by the technique used in this paper that the relations in $\mathrm{Ext}_R(\mathbf{k},\mathbf{k})$ are as follows

$$[Y_{a,q}, X_j] = \sum \lambda_{b,t} Y_{b,t}$$

where $\lambda_{b,t} = \pm 1$ or 0. Precisely, let $(a,q) = (a_1, \ldots, a_i; q_1, \ldots, q_{a_i})$, then

$$[Y_{a,q}, X_j] = -Y_{(a_1,\ldots,j,\ldots,a_i;\, q_1,\ldots,q_{a_i})} \quad \text{if } j \notin a,$$

$$[Y_{a,q}, X_j] = 0 \quad \text{if } j \in a \text{ and } j \neq a_i,$$

$$[Y_{a,q}, X_{a_i}] = \sum (-1)^{s+i} Y_{(a_1,\ldots,a_s,h,a_{s+1},\ldots,a_i,q_1,\ldots,q_h-1,\ldots,q_{a_i}+1)},$$

where the sum is over $h < a_i$, $h \notin a$, $q_h \geq 1$.

It is probable that there exists a nicer coordinate-free presentation of this Lie algebra (cf. the not particularly nice presentation of $\mathrm{sl}(n, \mathbf{R})$ by generators and relations).

References

[1] M. André, Hopf algebras with divided powers, J. Algebra 18 (1971) 19–50.
[2] E.S. Golod, On the homology of some local rings, Soviet Math. 3 (1962) 745–748.
[3] T.H. Gulliksen and G. Levin, Homology of Local Rings, Queens Papers in Pure Appl. Math. 20 (1969).
[4] C. Löfwall, On the subalgebra generated by the one-dimensional elements in the Yoneda Ext-algebra, in: J.-E. Roos, ed., Algebra, Algebraic Topology, and their Interactions, Lecture Notes in Math. (Springer, Berlin, to appear).
[5] S. MacLane, Homology (Springer, Berlin, 1967).
[6] W.S. Massey and F.P. Peterson, The cohomology structure of certain fibre spaces – I, Topology 4 (1965) 47–65.
[7] J.-P. Serre, Algèbre Locale: Multiplicités, Lecture Notes in Math. 11, 3rd ed. (Springer, Berlin, 1975).
[8] J. Shamash, The Poincaré series of a local ring, J. Algebra 12 (1969) 453–470.
[9] G. Sjödin, A set of generators for $\mathrm{Ext}_R(\mathbf{k}, \mathbf{k})$, Math. Scand. 38 (1976) 1–12.
[10] G. Sjödin, Hopf algebras and derivations, J. Algebra 64 (1980) 218–229.
[11] J. Tate, Homology of noetherian rings and local rings, Illinois, J. Math. (1957) 14–27.